

PERSISTENT BUNDLES OVER A TWO DIMENSIONAL COMPACT SET

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ABSTRACT. The C^1 -structurally stable diffeomorphisms of a compact manifold are those that satisfy Axiom A and the strong transversality condition (AS). We generalize the concept of AS from diffeomorphisms to invariant compact subsets. Among other properties, we show the structural stability of the AS invariant compact sets K of surface diffeomorphisms f . Moreover if \hat{f} is the dynamics of a compact manifold, which fibers over f and such that the bundle is normally hyperbolic over the non-wandering set of $f|_K$, then the bundle over K is persistent. This provides non trivial examples of persistent laminations that are not normally hyperbolic.

A classical result states that hyperbolic compact sets are C^1 -structurally stable. A compact subset K of a manifold M is *hyperbolic* for a diffeomorphism f of M if it is invariant ($f(K) = K$) and the tangent bundle of M restricted to K splits into two Tf -invariant subbundles contracted and expanded respectively. An invariant subset K of a diffeomorphism f is *C^1 -structurally stable* if every C^1 -perturbation f' of f lets invariant a compact set K' homeomorphic to K by an embedding C^0 -close to the inclusion $K \hookrightarrow M$ which conjugates the dynamics $f|_K$ and $f'|_{K'}$.

Such a result was generalized toward two directions that we would love to unify.

The first was to describe the *C^1 -structurally stable diffeomorphisms* of compact manifolds (K is then the whole manifold). This description used the so-called concept of *Axiom A diffeomorphisms*: the diffeomorphisms for which the non-wandering set is hyperbolic and equal to the closure of the set of periodic points. A diffeomorphism satisfies *Axiom A and the strong transversality condition* (AS) if moreover the stable and unstable manifolds of two non-wandering points intersect each other transversally.

The works of Smale [Sma67], Palis [PS70], de Melo [dM73], Mañé [Mañ88], Robbin [Rob71] and Robinson [Rob76] have achieved a satisfactory description of the C^1 -structurally stable diffeomorphisms stated in the following theorem.

Theorem 0.1 (Structural Stability). *The C^1 -structurally stable diffeomorphisms of compact manifolds are exactly the AS-diffeomorphisms.*

The concept of structural stability is fundamental since if one understands the global behavior of a structurally stable diffeomorphism, then he understands

the topological behavior of every perturbation of it. However such diffeomorphisms are not dense, and so this leads us to generalize this notion in order to include more diffeomorphisms. A first generalization is the Ω -stability: every perturbation of the dynamics has a homeomorphic non-wandering set and, via this homeomorphism the dynamics on the non-wandering set is conjugated. However Ω -stability does not imply that the interactions between the transitivity classes (*i.e.* basic pieces) are persistent and moreover is not generic in the C^2 topology (Newhouse phenomena). Also the Ω -stable diffeomorphisms are axiom A. Consequently every conservative Ω -stable diffeomorphism is Anosov (the whole manifold is hyperbolic). This reduces a lot the variety of the examples.

This is why we shall generalize the concept of AS from diffeomorphisms to invariant compact subsets. An invariant, compact subset K of a C^1 diffeomorphism is AS if:

- (i) The intersection Λ of the non-wandering set with K is hyperbolic and is contained in the closure of the periodic points,
- (ii) there exists $\epsilon > 0$ such that for every points $x, y \in K$ the ϵ -local stable manifold $W_\epsilon^s(x)$ intersects transversally $W_\epsilon^u(y)$ and the intersection is contained in K .

We recall that the ϵ -local stable manifold of x is $W_\epsilon^s(x) := \{y : d(f^n(x), f^n(y)) < \epsilon, \forall n \geq 0\}$.

Example 0.2. Let f be a diffeomorphism which lets invariant two hyperbolic compact subsets K_1 and K_2 . We suppose that K_1 and K_2 are included in the closure of the periodic points of f . Let $(W_{loc}^s(x))_{x \in K_1}$ be a continuous family of local stable manifolds of points of K_1 ; let $(W_{loc}^u(y))_{y \in K_2}$ be a continuous family of local unstable manifolds of points of K_2 . We suppose that the intersection of $W_{loc}^s(x)$ with $W_{loc}^u(y)$ is transverse and compact for all $x \in K_1$ and $y \in K_2$. Then the union $K_{12} := \cup_{(x,y) \in K_1 \times K_2} W_{loc}^s(x) \cap W_{loc}^u(y)$ is a compact subset. Also $K := K_1 \cup K_2 \cup \bigcup_{n \in \mathbb{Z}} f^n(K_{12})$ is an AS compact subset.

We notice that K is not hyperbolic if the dimension of the stable directions of K_1 and K_2 are different. This the case for perturbations of the conservative dynamics of the product of the Riemannian sphere with the real line $(z, t) \mapsto (2z, \frac{1+2z}{2+z}t)$. For many perturbations f , the hyperbolic fixed point K_1 close to $(0, 0)$ has a local stable manifold that intersects transversally an unstable local manifold of the hyperbolic fixed point K_2 close to $(\infty, 0)$ at a circle K_{12} .

We will generalize in section 1.1 some of the dynamical properties of AS diffeomorphisms to AS compact subsets.

This is a first theorem of this paper:

Theorem 0.3. *If K is an AS compact subset for a C^1 -diffeomorphism a manifold M of dimension at most 2, then K is structurally stable.*

Example 0.4. Let f be a Morse-Smale diffeomorphism of a surface M . This means that f is an AS diffeomorphism with a finite non wandering set.

Consequently there exists an attracting periodic orbit $(p_i)_{i=1}^n$. We suppose that the eigenvalues of the derivative of f^n at this orbit are not real.

As an algebraic geometer we blow up each point p_i to a circle \mathbb{S}_i and a neighborhood of it to a Möbius strip. This makes a new surface \tilde{M} . The dynamics f lifts to a dynamics \tilde{f} on \tilde{M} which preserves the circle and acts on them as a rotation.

Moreover, the manifolds $\tilde{M} \setminus \cup_i \mathbb{S}_i$ and $M \setminus \cup_i \{p_i\}$ are equal, also the corresponding restrictions of the dynamics \tilde{f} and f are equal. Therefore, the complement of the attraction basin of $\cup_i \mathbb{S}_i$ is an AS compact subset of \tilde{M} .

Another generalization of the structural stability theorem of a hyperbolic set K is to blow up to a manifold every point of K , in order to obtain a family of disjoint immersed manifolds which depend locally C^1 continuously. This reaches the concept of lamination. A *lamination* is a metric space L locally modeled on the product of \mathbb{R}^n with a locally compact metric space T such that the changes of coordinates preserve the horizontal bundle and are continuously differentiable along the fibers on all the domain. A *plaque* is a set of the form $\phi^{-1}(\mathbb{R}^n \times \{t\})$ where ϕ is a chart. The *leaf* of $x \in L$ is the union of all the plaques which contain x . A maximal atlas \mathcal{L} of compatible charts is a *lamination structure* on L . Given an open subset L' of L , we denote by $\mathcal{L}|_{L'}$ the structure of lamination on L' formed by the charts of \mathcal{L} whose domain is included in L' . The reader not familiar with the laminations should look at [Ghy99], [Ber08a].

Laminations are specially interesting when they are embedded. An *embedding* of a lamination (L, \mathcal{L}) into a manifold M is a homeomorphism i onto its image which is an *immersion*. A continuous map $i: L \rightarrow M$ is an *immersion* if its differential along the plaques of \mathcal{L} exists, is injective and depends continuously on $x \in L$. Two embeddings are C^1 -close if they are close in the C^0 -compact-open topology and their differential along the plaques are close for the compact-open topology. The reader might see [Ber08a] for more details about this topology.

Usually we identify an embedded lamination with its image. We note that its plaques are submanifolds of M and its leaves form a family of disjoint injectively immersed submanifolds. Thus the tangent space $T_x \mathcal{L}$ at $x \in L$ of its leaf is identified to a subspace of the tangent space $T_x M$ of M at x . A diffeomorphism f of M *preserves* this lamination if it sends each leaf of \mathcal{L} into a leaf of \mathcal{L} , or equivalently each plaque of \mathcal{L} into a plaque of \mathcal{L} . Such a lamination is C^1 -persistent if for f' C^1 close to f there exists an embedding i' C^1 -close to i such that f' preserves \mathcal{L} embedded by i' and induces the same dynamics on the leaves of \mathcal{L} as f . We notice that when the *dimension* of the lamination (that is of its leaves) is zero, then the lamination \mathcal{L} is persistent *iff* the subset L is structurally stable.

The founders of this way of generalization were Hirsch, Pugh and Shub (HPS). This is their main theorem:

Theorem 0.5 (Hirsch-Pugh-Shub [HPS77]). *Normally hyperbolic and plaque-expansive laminations are persistent.*

This theorem is the fundamental one of the partially hyperbolic dynamics field.

The plaque-expansiveness is a generalization of the expansiveness to the space of leaves. The necessity and automaticity of the plaque-expansiveness in the above theorem is still an open problem. We will give its definition in Section 2.1.

Let us recall that a diffeomorphism f is *normally hyperbolic* to an embedded lamination (L, \mathcal{L}) if the tangent bundle of M restricted to L is the direct sum of $T\mathcal{L}$ with two Tf invariant subbundles $E^s \rightarrow L$ and $E^u \rightarrow L$ such that the following property holds. There exists $\lambda < 1$ such that for every $x \in L$, for every $n \geq 0$ sufficiently large, for all unit vectors $v_c \in T_x \mathcal{L}$, $v_s \in E_x^s$ and $v_u \in E_x^u$:

$$\|T_x f^n(v_s)\| < \lambda^n \min(1, \|T_x f^n(v_c)\|) \quad \text{and} \quad \lambda^n \|T_x f^n(v_u)\| > \max(1, \|T_x f^n(v_c)\|).$$

When L is compact, n can be chosen independently of x . Otherwise this is in general not the case. If the strong stable direction E^s is 0, then the lamination (L, \mathcal{L}) is *normally expanded* by f .

A paradigmatic application of the HPS's theorem is when f is the product dynamics of an Anosov diffeomorphism of a compact manifold M with the identity of a compact manifold N . Then *the bundle $M \times N \rightarrow M$ is C^1 -persistent as a lamination*. This means that for every f' C^1 -close to f , there exists a continuous family of disjoint C^1 -submanifolds $(N'_x)_{x \in M}$ s.t. N'_x is C^1 -close to $\{x\} \times N$ and $f'(N'_x) = N'_{f'(x)}$.

Let us generalize the above example by considering that f is the product dynamics of an AS diffeomorphism g of M with the identity of a manifold N . Then the persistence of \mathcal{L} is not a consequence of the HPS' theorem if g is not Anosov nor a consequence of the structural stability theorem if N has non-zero dimension. However it is the consequence of the main theorem of this paper if the dimension of M is 2 (the case of dimension 1 is easy).

Theorem 0.6 (Main result). *Let $\pi: \hat{M} \rightarrow M$ be a C^1 -bundle over a surface with compact fibers. Let K be an AS compact subset for a diffeomorphism f of M . Let \hat{f} be a diffeomorphism of \hat{M} which preserves the bundle π and lift f . We suppose that the bundle is normally hyperbolic over the intersection between K and the non-wandering set of f . Then the bundle over K is C^1 -persistent. In other words the lamination (L, \mathcal{L}) supported by $\pi^{-1}(K)$ and whose leaves are the connected component of fibers of K 's points is C^1 -persistent.*

A simple application of this theorem is when K is equal to M and so f is an AS diffeomorphism of a surface; \hat{f} is the product dynamics on $\hat{M} = M \times N$ of f with the identity on a compact manifold N .

Example 0.7. Let us come back to the dynamics $f: (z, t) \mapsto (2z, \frac{1+2z}{2+z}t)$ of the product of the sphere with the real line in Example 0.2. By using Theorem 0.6, the canonical bundle by line neutralized and then compactified

at the infinity is persistent. This dynamics appears in some hetero-dimensional cycles of dynamics of \mathbb{R}^3 . For instance when the cycle is supported by the union of the unit sphere with the vertical line passing through the poles $\{0, \infty\}$ of the sphere, and whose dynamics at the neighborhood of unit sphere is f . The persistence of this bundle might be useful for showing some non uniform hyperbolic properties of perturbations of this hetero-dimensional cycle.

We notice that contrarily to the previous theorems, the hypothesis of this one is not open. Actually the proof will exhibit a class of laminations which is open, but too complicated to be described here. Though the above theorem generalizes the structural stability theorem in dimension two, it does not generalize the HPS theorem. In order to state a conjecture generalizing both results, let us recall the definition of a saturated set.

A *saturated set* Λ of a lamination (L, \mathcal{L}) embedded into a manifold M is a union of leaves of \mathcal{L} . If K is subset of L , the \mathcal{L} -saturated set of K is the union of the leaves of K 's points. We note that if K is an invariant compact subset and if L is compact, then its \mathcal{L} -saturated set is an invariant compact subset.

A compact lamination (L, \mathcal{L}) embedded into a manifold M and preserved by a diffeomorphism f is *normally AS* if there exists $\epsilon > 0$ such that:

- The saturated $\Omega(\mathcal{L})$ of the non-wandering set of f restricted to L is normally hyperbolic (and plaque-expansive),
- the ϵ -local stable set of a leaf of $\Omega(\mathcal{L})$ (which is an immersed manifold) intersects transversally the ϵ -local unstable set of every leaf of $\Omega(\mathcal{L})$,
- $\Omega(\mathcal{L})$ is locally maximal.

This is our conjecture:

Conjecture 0.8. *Compact normally AS laminations are C^1 -persistent.*

We notice that the main theorem solves this conjecture when the leaves of the foliation are the connected components of a preserved C^1 -bundle over a surface.

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1. GEOMETRY AND DYNAMICS OF AS COMPACT SUBSETS

Axiom A and AS diffeomorphisms were deeply studied in the 60-70's. We are going to generalize some of these results to AS compact sets.

Let f be a diffeomorphism of a compact manifold M and let K be an AS compact subset f -invariant. Let Λ be the non-wandering set of the restriction of f to K .

For $x \in \Lambda$, we denote by $W_K^s(x)$ the intersection of the stable manifold of x with K ; we denote by $W_{K^\epsilon}^s(x)$ the union of the ϵ -local stable manifolds of points of $W_K^s(x)$. We define similarly $W_K^u(x)$ and $W_{K^\epsilon}^u(x)$.

Let $\lambda < 1$ be greater than the contraction of the stable direction of Λ . Let $\tilde{d}: (x, y) \mapsto \sup_{n \geq 0} \max \left(\frac{d(f^n(x), f^n(y))}{\lambda^n}, 1 \right)$. For $x \in K$ and $\epsilon > 0$, let $\tilde{W}_\epsilon^s(x)$ be the ball centered at x and with radius ϵ for this metric \tilde{d} . We notice that this ball is a local stable manifold of x . Moreover f sends the closure of $\tilde{W}_\epsilon^s(x)$ into $\tilde{W}_\epsilon^s(f(x))$. We call $\tilde{W}_\epsilon^s(x)$ the *adapted ϵ -local stable manifold of x* .

1.1. Dynamics on K . If for two periodic points $x, y \in \Lambda$, the set $W_K^s(x)$ intersects $W_K^u(y)$ and $W_K^s(y)$ intersects $W_K^u(x)$, we note $x \sim y$. This relation is obviously reflexive and symmetric. Let us show its transitivity.

First we note that two points in relation \sim have their stable manifold (resp. unstable manifold) of the same dimension. Moreover, the relation is f -invariant: $x \sim y$ iff $f(x) \sim f(y)$.

Let x, y and z be three periodic points of Λ such that $x \sim y$ and $y \sim z$. By the f -invariance of \sim , the points x and z are in relation for f if they are in relation for any iterate of f . So we can suppose that x, y and z are fixed points of f . Let y_x and y_z be two points of the intersections $W_K^s(x) \cap W_K^u(y)$ and $W_K^s(y) \cap W_K^u(z)$ respectively. By iterating the dynamics, we may assume that y_x and y_z are close to y . By the λ -lemma, $W_\epsilon^s(y_x)$ and $W_\epsilon^u(y_z)$ are close to $W_\epsilon^s(y)$ and $W_\epsilon^u(y)$ respectively. Thus the intersection $W_\epsilon^s(y_x) \cap W_\epsilon^u(y_z) \subset W_{K\epsilon}^s(x) \cap W_{K\epsilon}^u(z)$ is not empty. And so by Property *ii* of the *AS* definition, the intersection $W_K^u(x) \cap W_K^s(z)$ is non empty. Such an argument proves that $x \sim z$. Thus the relation \sim is an equivalent relation.

As two close points of $\Lambda \cap \text{Per}(f)$ are equivalent, the sets supporting the different equivalent classes of \sim are δ -distant for some $\delta > 0$.

We denote by $(\Lambda_{i,j})_{i,j}$ the closure of the equivalent classes in Λ . They are δ -distant and their union is equal to the compact set Λ , by density of the periodic points. Thus there are finitely many equivalent classes. By f invariance of \sim , these equivalent classes are all periodic. We put $f(\Lambda_{i,j}) = \Lambda_{i,j+1}$ and $\Lambda_i := \cup_j \Lambda_{i,j}$.

The family $(\Lambda_i)_i$ is the *spectral decomposition* of Λ and each of its element is a *basic set*.

Let us show that each basic set Λ_i is transitive: for all $x, y \in \Lambda_i$, for all neighborhoods U of x and V of y for the topology of K , there exists $n > 0$ such that $f^n(U) \cap V$ is nonempty. By density of the periodic points, we only need to show this when x and y are periodic. By replacing y by some of its iterates, we may suppose that x and y belong to a same $\Lambda_{i,j}$. Let $q \in W_K^u(x) \cap W_K^s(y)$. For every sufficiently large multiple n of the period of x and y , $f^{-n}(q)$ belongs to U and $f^n(q)$ belongs to V . Indeed $f^{2n}(U)$ intersects V .

As each basic set Λ_i is hyperbolic and in the closure of the periodic orbits, the stable set of Λ_i :

$$W^s(\Lambda_i) := \{x \in M : d(f^n(x), \Lambda_i) \rightarrow 0, \text{ when } n \rightarrow +\infty\}$$

is equal to the union of manifolds $\cup_{x \in \Lambda_i} W^s(x)$. The reader should look at [Shu78], Prop 9.1 for a proof.

Intersecting with K , we get:

$$W_K^s(\Lambda_i) := \{x \in K : d(f^n(x), \Lambda_i) \rightarrow 0, \text{ when } x \rightarrow +\infty\} = \bigcup_{x \in \Lambda_i} W_K^s(x).$$

Let us write $\Lambda_i \succ \Lambda_j$ if $W_K^u(\Lambda_i) \setminus \Lambda_i$ intersects $W_K^s(\Lambda_j) \setminus \Lambda_j$.

Proposition 1.1. *If $\Lambda_i \succ \Lambda_j$ then there exist two periodic points $p_i \in \Lambda_i$ and $p_j \in \Lambda_j$ such that $W^u(p_i)$ intersects $W^s(p_j)$.*

Proof. Let $q_i \in \Lambda_i$ and $q_j \in \Lambda_j$ be two points such that $W^u(q_i)$ intersects $W^s(q_j)$ at some point $q \in K$. Thus for $n > 0$ large, the iterate $f^n(q)$ is close to $f^n(q_j)$. Consequently the ϵ -local unstable manifold of $f^n(q)$ intersects the ϵ -local stable manifold of a periodic point $f^n(p_j) \in \Lambda_j$ close to $f^n(q_j)$. Let $f^n(p')$ be this intersection point. It must belong to K . Pulling back this construction by f^n , we get the existence of an intersection point $p' \in K$ between $W^s(q_i)$ and $W^u(p_j)$. We construct similarly p_i . \square

Proposition 1.2. *The relation \succ can be completed to a total order.*

Proof. This proposition is equivalent to the no-cycle condition: if $\Lambda_{j+1} \succeq \Lambda_j$ for $j \in \{1, \dots, n\}$ with $\Lambda_{n+1} = \Lambda_1$, then $\Lambda_j = \Lambda_{j+1}$ for every $j \in \{1, \dots, n\}$. Let us construct by induction on i , a periodic point $p_i \in \Lambda_i$ such that $W_K^u(p_i)$ intersects $W_K^s(p_{i+1})$ at a point $q_i \in K$ and for $i = n$, $f^k(p_{n+1}) \sim p_1$ for some k .

The existence of p_1 and p_2 follows from the last proposition. Let us suppose $(p_l)_{l \leq i}$ constructed. There exist periodic points $p'_i \in \Lambda_i$ and $p_{i+1} \in \Lambda_{i+1}$ such that $W_K^u(p'_i)$ intersects $W_K^s(p_{i+1})$. By replacing the points p'_i and p_{i+1} by their image by some iterate f^k , we may assume that $p_i \sim p'_i$. Thus, by proceeding as for the proof of the transitivity of \sim , we get that $W_K^u(p_i)$ intersects $W_K^s(p_{i+1})$. The last statement for $i = n$ is clear.

One easily shows that each point $q_i \in W_K^u(p_i) \cap W_K^s(p_{i+1})$ is non-wandering and so belongs to Λ . Looking at the backward and forward orbit of q_i , this point belongs to a basic piece which is arbitrarily close to Λ_i and Λ_{i+1} . So indeed the $(\Lambda_i)_i$ coincide. \square

We renumber the sets $(\Lambda_i)_i$ according to this total order, so $\Lambda_i \prec \Lambda_j$ if and only if $i < j$. We denote by $\text{int}^K(E)$ the interior of a subset E of K for the topology induced by K .

Theorem 1.3. *Let f be a homeomorphism of a compact metric space K . Let $\Lambda = \Lambda_1 \coprod \dots \coprod \Lambda_N$ be a disjoint union of invariant closed subsets which contains the limit set. If for every $i < j$, $W_K^u(\Lambda_i) \cap W_K^s(\Lambda_j) = \emptyset$ then there exists of a sequence of closed subsets of K :*

$$\emptyset = K_0 \subset K_1 \subset \dots \subset K_N = K$$

such that for all $i \geq 1$:

- (1) $f(K_i) \subset \text{int}^K(K_i)$,
- (2) $\Lambda_i = \bigcap_{n \in \mathbb{Z}} f^n(K_i \setminus K_{i-1})$.

Such a sequence $(K_i)_i$ is called a *filtration adapted to* $(\Lambda_i)_i$.

For a proof the reader should read Theorem 2.3 of [Shu78]. Actually the cited theorem asks K to be a manifold, but this fact is useful uniquely to chose $(K_i)_i$ among the submanifolds with boundary. This is not asked here.

1.2. Geometrical structures on a local stable set of K . We are going to endow a local stable set of K with a structure of stratification of laminations.

Following J. Mather [Mat73], a *stratified space* is a metric space A equipped with a finite partition Σ of A into locally closed subsets, satisfying the axiom of the frontier:

$$\forall (X, Y) \in \Sigma^2, \text{cl}(X) \cap Y \neq \emptyset \Rightarrow \text{cl}(X) \supset Y. \quad \text{We note then } X \geq Y.$$

The pair (A, Σ) is called *stratified space* with *support* A and *stratification* Σ .

A *stratification of laminations* is the data of a lamination structure on each stratum of a stratification, such that for any strata $X \geq Y$, the dimension of X is at least equal to the dimension of Y .

Let us work under the hypotheses of main Theorem 2.2. We recall that $\pi: \hat{M} \rightarrow M$ is the bundle that \hat{f} preserves over f . Let $(\Lambda_i)_i$ be the spectral decomposition of $f|_K$. Let $(K_i)_i$ be an adapted filtration to $(\Lambda_i)_i$.

When K is equal to the whole manifold M , we have proved in Proposition 1.2.7 of [Ber07] that each stable set $X_i := W^s(\Lambda_i)$ has a lamination structure whose leaves are the local stable manifolds of Λ_i 's points. Moreover the laminations $(X_i)_i$ form the strata of a stratification of laminations Σ on $K = M$ ¹. Moreover $X_i \leq X_j$ is equivalent to $\Lambda_i \succeq \Lambda_j$ and so implies $i \geq j$.

Pulling back this construction by π , we get on \hat{M} a stratification $\hat{\Sigma}$ whose strata are $\hat{X}_i := \pi^{-1}(X_i)$ and whose leaves are the connected components of the preimages by π of the leaves of X_i .

In order to prove the persistence of these stratifications of laminations, we will need the existence of trellis structures on (A, Σ) and on $(\hat{A}, \hat{\Sigma})$.

Definition 1.4. A *trellis (of laminations)* on a laminar stratified space (A, Σ) is a family of laminations $\mathcal{T} := (L_X, \mathcal{L}_X)_{X \in \Sigma}$ such that for all strata $X \leq Y \in \Sigma$:

- L_X is a neighborhood of X and each leaf of X is a leaf of \mathcal{L}_X ,
- each plaque P of \mathcal{L}_Y included in L_X is C^1 -foliated by plaques of \mathcal{L}_X ,
- every point, close to P , is included in a plaque P' of \mathcal{L}_Y included in L_X , and whose foliation by \mathcal{L}_X -plaques which is diffeomorphic and C^1 -close to the one of P .

The lamination (L_X, \mathcal{L}_X) is *the tubular neighborhood of X* .

We notice that the tubular neighborhoods of the 0-dimensional strata are just laminations by points on a neighborhood of them. The tubular neighborhoods

¹Here the dimension of M is irrelevant.

of the strata of maximal dimension are equal to their strata since they are open. In case Σ is the canonical stratification induced by an AS diffeomorphism, the 0-dimensional strata correspond to repulsive periodic orbits. The maximal strata correspond to the basin of the attracting basic sets, laminated by stable manifolds.

To construct the other tubular neighborhoods, here it is much simpler to assume that M is a surface. We can use the following result of de Melo:

Proposition 1.5 ([dM73], Thm 2.2 with its Rmk above). *Let f be an AS diffeomorphism of a compact surface. Let X be the stable set of the union of basic sets with a one dimensional stable direction. Then X can be endowed with the structure of lamination whose leaves are the stable manifolds. Moreover there exists a lamination \mathcal{L}_X on a neighborhood L_X of X such that every leaf of X is a leaf of \mathcal{L}_X and the leaves of \mathcal{L}_X that do not intersect X form a C^1 foliation on $L_X \setminus X$.*

This proposition provides the last tubular neighborhoods of the stratification Σ : we give the same tubular neighborhood (L_X, \mathcal{L}_X) for all the basic sets with one dimensional stable direction.

Let us construct similar structures for AS compact subsets of a surface M .

Proposition 1.6. *Let f be a diffeomorphism of a surface M . Let K be an AS compact subset of M . Then for every small $\epsilon > 0$, there exists a stratification of laminations (A, Σ) endowed with a trellis structure \mathcal{T} such that:*

- *The support A is equal to $\cup_{x \in K} \tilde{W}_\epsilon^s(x)$.*
- *Each stratum X_k of Σ is associated to a basic piece Λ_k . The support of X_k is $\tilde{W}_{K\epsilon}^s(\Lambda_k) := \cup_{x \in W_K^s(\Lambda_k)} \tilde{W}_\epsilon^s(x)$. The plaques of X_k are local stable manifolds.*
- *Each support of tubular neighborhood L_{X_k} is $f|_A^{-1}$ -stable: the preimage by $f|_A$ of L_{X_k} is included in L_{X_k} .*
- *Each lamination \mathcal{L}_{X_k} is locally f -invariant: every plaque of \mathcal{L}_{X_k} contained in $f^{-1}(L_{X_k})$ is sent by f into a plaque of \mathcal{L}_{X_k} .*
- *The tubular neighborhoods are compatible: for every $x \in L_{X_k} \cap L_{X_j}$ with $j \leq k$, every plaque of \mathcal{L}_{X_k} containing x is included in a plaque of \mathcal{L}_{X_j} .*

We notice that the closure of A is sent by f into A and that A contains K .

Let \hat{X}_k and $(L_{\hat{X}_k}, \mathcal{L}_{\hat{X}_k})$ be the laminations whose leaves are the connected components of preimages by π of the leaves of X_k and $(L_{X_k}, \mathcal{L}_{X_k})$ respectively. Let $\hat{A} := \pi^{-1}(A)$. We notice that $(\hat{A}, \hat{\Sigma})$ is a stratified space endowed with the trellis structure $\hat{\mathcal{T}} := (L_{\hat{X}_k}, \mathcal{L}_{\hat{X}_k})_k$.

Proof. Let $(K_j)_{j=1}^N$ be an adapted filtration to the spectral decomposition $(\Lambda_j)_{j=1}^N$ of K . The proposition is a consequence of the following induction hypothesis on $j \in \{1, \dots, N\}$.

For every sufficiently small $\epsilon > 0$, there exists a stratification of laminations (A_j, Σ_j) endowed with a trellis structure \mathcal{T}_j such that:

- The support of A_j is equal to $\cup_{x \in \text{int}^K(K_j)} \tilde{W}_\epsilon^s(x)$.
- Each stratum of X_k^j contains a unique basic piece Λ_k , for $k \leq j$. The support is

$$X_k^j := \bigcup_{x \in \text{int}^K(K_j) \cap W^s(\Lambda_k)} \tilde{W}_\epsilon^s(x).$$

The plaques of X_k^j are local stable manifolds.

- For each $k \leq j$, the support $L_{X_k^j}$ is $f|_{A_j}^{-1}$ -stable and the lamination $\mathcal{L}_{X_k^j}$ is locally invariant.
- The tubular neighborhoods of \mathcal{T}_j are compatible.

We notice that the closure of A_j is sent by f into A_j . Moreover A_j contains K_j .

Step $j = 1$. We put $L_1^1 := A_1^1$.

If the dimension of the stable direction of Λ_1 is equal to 0 or 2, then Λ_1 is a repulsive or attracting periodic orbit. Let $X_1^1 = (L_1^1, \mathcal{L}_1^1)$ be the lamination whose leaves are the connected components of $A_1 = L_1^1$.

If the dimension of the stable direction of Λ_1 is equal to 1, then for ϵ small enough, the union of local stable manifolds $W_\epsilon^s(\Lambda_1) := \cup_{x \in \Lambda_1} W_\epsilon^s(x)$ has a lamination structure whose plaques are local stable manifolds.

As $f^n(A_1^1)$ is a subset of $W_{f^n(K_1)\lambda^{n\epsilon}}^s(\Lambda_1)$, for n sufficiently large, A_1^1 is sent into $W_\epsilon^s(\Lambda_1)$ by f^n . Also $f^n(A_1^1)$ is open in $W_\epsilon^s(\Lambda_1)$. For let $x \in f^n(A_1^1)$ and $x' \in W_\epsilon^s(\Lambda_1)$ close to x . Let $z \in f^n(\text{int}^K K_1)$ be such that x belongs to $f^n(\tilde{W}_\epsilon^s(f^{-n}(z))) \subset W_\epsilon^s(z)$ and let $z' \in \Lambda_1$ be s.t. x' belongs to $W_\epsilon^s(z')$. Thus for x' close enough to x , $W_\epsilon^u(z)$ intersects $W_{2\epsilon}^s(z')$ at a point y close to z . The point y must belong to K by property *ii* of AS compact set. As $\text{int}^K K_1$ is open, for x' close enough to x , the point y belongs to $f^n(\text{int}^K K_1)$. As moreover y is close to z , by continuity of the adapted local stable manifolds on $W_K^s(\Lambda_1)$, the point x' belongs to $f^n(\tilde{W}_\epsilon^s(f^{-n}(y)))$. Hence x' belongs to $f^n(A_1)$.

We endow $L_1^1 = A_1$ with the lamination structure \mathcal{L}_1^1 given by the pull back by f^n of the one of $W_\epsilon^s(\Lambda_1)$ restricted to the open subset $f^n(A_1)$. We put $X_1^1 = (L_1^1, \mathcal{L}_1^1)$.

Step $j \rightarrow j + 1$. Let us assume the induction hypothesis for $j \geq 1$.

If the dimension of the stable direction of Λ_{j+1} is equal to 2, then we proceed as above since Λ_{j+1} is minimal for \prec .

Let us suppose that the dimension of the stable direction is 0. Note that $A_{j+1} := \cup_{n \geq 0} f|_{A_{j+1}}^{-n}(A_j) \cup \Lambda_{j+1}$. Moreover as the closure of A_j is sent by f into A_j , the closure of $f|_{A_{j+1}}^{-n-1}(A_j)$ is sent by f^2 into $f|_{A_{j+1}}^{-n}(A_j)$.

We put $X_{j+1}^{j+1} := \Lambda_{j+1}$ the lamination whose leaves are the points of the periodic orbit. Then we pull back canonically the stratification Σ_j on $A_{j+1} \setminus \Lambda_{j+1}$ by the dynamics $f|_{A_{j+1}}$; this forms with X_{j+1}^{j+1} a family of laminations Σ_{j+1} on A_{j+1} .

The frontier condition of this partition between all the strata but X_{j+1}^{j+1} follows from the one of Σ_j ; the frontier condition with X_{j+1}^{j+1} is obvious since this last stratum is a periodic orbit. Thus Σ_j is a stratification of laminations.

The requested trellis structure \mathcal{T}_{j+1} is constructed by pulling back the tubular neighborhoods of the strata of Σ_i and by endowing X_{j+1} with a 0-dimensional tubular neighborhood.

Otherwise the dimension of the stable direction is equal to 1. Let us recall a few definitions.

A *fundamental domain* for the unstable manifolds of a hyperbolic set Λ , is a closed subset $D \subset W^u(\Lambda)$ such that

$$\bigcup_{n \in \mathbb{Z}} f^n(D) \supset W^u(\Lambda) \setminus \Lambda.$$

The fundamental domain D is *proper* if moreover $D \cap \Lambda$ is empty. A *fundamental neighborhood* for the unstable manifolds, N^s , is a neighborhood of a fundamental domain such that $N \cap \Lambda$ is empty.

Let us recall the

Lemma 1.7 ([HPPS70] Lemma 3.2). *Let Λ be a hyperbolic set for f included in the closure of the periodic points. For every δ small enough:*

$$\Delta := cl\left(W_\delta^u(\Lambda) \setminus f^{-1}(W_\delta^u(\Lambda))\right)$$

is a proper fundamental domain for $W^s(\Lambda)$.

Applying this lemma to the basic set Λ_{j+1} that we study, for $\epsilon > \delta > \delta' > 0$ small enough the following set:

$$\Delta := W_\delta^u(\Lambda_{j+1}) \setminus cl\left(f^{-1}(W_{\delta'}^u(\Lambda_{j+1}))\right).$$

is a proper fundamental domain for $W^u(\Lambda_{j+1})$. Moreover the intersection $D := \Delta \cap K$ is included in K_{j+1} .

By the filtration properties, there exists $M > 0$, such that $D := \Delta \cap K$ is sent by f^M into $K_j \subset A_j$. For ϵ small enough, the one dimensional family of unstable local manifolds:

$$\mathcal{F}_u := \{W_\epsilon^u(x); x \in f^m(D), 1 \leq m \leq M\}$$

is transverse to the ϵ -stable manifolds of the points of the sets $(W_K^u(\Lambda_i))_{i \leq j}$ and so to the laminations of Σ_j . By compactness of $\bigcup_{m=1}^M f^m(D)$, we can restrict the support of the tubular neighborhoods forming the trellis structure \mathcal{T}_j , so that the leaves of each $\mathcal{L}_{X_k^j}$ are transverse to the family of unstable local manifolds \mathcal{F}_u . In order to reduce and preserve the f^{-1} -stability of the tubular neighborhoods, we can take away from $L_{X_k^j}$ the support of the strata lower than X_k^j and the preimage $f^{-n}(\tilde{W}_\epsilon^s(K_{k-1}))$, for some large n , with $\tilde{W}_\epsilon^s(K_{k-1}) = \bigcup_{x \in K_{k-1}} \tilde{W}_\epsilon^s(x)$.

The union of the adapted local stable manifolds $D_\epsilon := \cup_{x \in D} \tilde{W}_\epsilon^s(x)$ is a fundamental open neighborhood for the unstable manifolds in the topology of A_{j+1} . This follows from the fact that every point of $A_{j+1} \setminus X_{j+1}^{j+1}$ belongs to the stable set of a lower basic piece. By the strong transversality condition and the existence of a trellis structure \mathcal{T}_{j+1} , a point $x \in A_{j+1} \setminus X_{j+1}^{j+1}$ close to a point of D must have its adapted ϵ -local stable manifold that intersects $W_\delta^u(\Lambda_j) \subset W_\epsilon^u(\Lambda_j)$. By property *ii* of the *AS* compact subsets, this intersection is included in K and so x belongs to D_ϵ .

The set $f^M(D_\epsilon) \cap A_j$ is an open subset of A_j and so can be stratified by the restriction of Σ_j to $f^M(D_\epsilon) \cap A_j$. Let X and $(\tilde{L}, \tilde{\mathcal{L}})$ be the pull backs by f^M of the union of the one dimensional strata X_k^j of Σ_j and their tubular neighborhoods $(L_{X_k^j}, \mathcal{L}_{X_k^j})$ of \mathcal{T}_j restricted to $f^M(D_\epsilon) \cap L_{X_k^j}$. We notice that $(\tilde{L}, \tilde{\mathcal{L}})$ is transverse to $W_\epsilon^u(\Lambda_{j+1})$.

We remark that \tilde{L} is an open neighborhood of \tilde{X} in D_ϵ . Also $(\tilde{L}, \tilde{\mathcal{L}})$ is locally f -invariant. Moreover the restriction $(\tilde{L} \setminus \tilde{X}, \tilde{\mathcal{L}}|_{\tilde{L} \setminus \tilde{X}})$ is a C^1 -foliation on an open subset of M and $D_\epsilon \setminus \tilde{X}$ is open in M .

We want to extend $(\tilde{L} \setminus \tilde{X}, \tilde{\mathcal{L}}|_{\tilde{L} \setminus \tilde{X}})$ to a C^1 -foliation on $D_\epsilon \setminus \tilde{X}$ transverse to \mathcal{F}_u and locally f -invariant. As the leaves of $\tilde{\mathcal{L}}$ are of dimension one, this would be simple if this foliation would have been smooth. However it is in general not the case since f is of class C^1 . The construction is done in three steps:

- First we suppose δ small enough (and then δ' small enough) in order that $T\tilde{\mathcal{L}}$ can be continuously extended to a continuous line field χ on an open neighborhood of D_ϵ , uniformly transverse to \mathcal{F}_u .
- We restrict $(\tilde{L}, \tilde{\mathcal{L}})$ (by restricting the tubular neighborhoods of the higher strata) in order that this lamination can be smoothly extended on a neighborhood of $D_\epsilon \cap cl(\tilde{L}) \setminus \tilde{X}$ in D_ϵ by the following lemma:

Lemma 1.8 (Lemma 1.4 [dM73]). *Let \mathcal{F} be a one dimensional C^1 -foliation of an open subset U of a manifold. Let C be compact subset of U and $\eta > 0$. Then for every neighborhood \hat{C} of C there exists a foliation \mathcal{F}' which coincides with \mathcal{F} on a neighborhood of C , which is smooth on the complement of \hat{C} and whose tangent space is everywhere η -close to the one of \mathcal{F} .*

Then we patch this smooth extension to a smooth approximation $\tilde{\chi}$ of χ in order to form a lamination \mathcal{F} on D_ϵ such that:

- \mathcal{F} restricted to \tilde{L} is $\tilde{\mathcal{L}}$,
- \mathcal{F} restricted to $D_\epsilon \setminus \tilde{L}$ is smooth.
- For δ' sufficiently close to δ , $B^+ := \tilde{W}^s(D^+)$ is disjoint from $B^- := \tilde{W}^s(D^-)$, with $D^+ := K \cap W_\delta^u(\Lambda_{j+1}) \setminus cl(W_{\delta'}^u(\Lambda_{j+1}))$ and $D^- := f^{-1}(D^+)$. To make \mathcal{F} locally f -invariant, we proceed as above by substituting $(\tilde{L}, \tilde{\mathcal{L}})$

to \mathcal{F} restricted to $B^+ \cup B^- \cup \tilde{L}$. This makes a lamination $\hat{\mathcal{L}}$ on D_ϵ which is a C^1 -foliation restricted to $D_\epsilon \setminus \tilde{X}$ and locally f -invariant. Moreover we can suppose all the approximations done sufficiently small such that $\hat{\mathcal{L}}$ is transverse to the ϵ -local unstable manifold of Λ_{j+1} 's points.

As in the case $j = 1$, the set $\cup_{x \in W^s(\Lambda_{j+1}) \cap \text{int}^K(K_{j+1})} \tilde{W}_\epsilon^s(x)$ supports a canonical lamination X_{j+1}^{j+1} whose plaques are local stable manifolds. As before again, the strata of Σ_i pull back along the orbit of $f|_{A_{j+1}}$ to form a stratification of laminations on $A_{j+1} \setminus X_{j+1}^{j+1}$. Let Σ_{j+1} be the union of this stratification with X_{j+1} .

The frontier condition is clear between all the laminations $(X_i^{j+1})_{i \leq j}$. Let us prove it for X_{j+1}^{j+1} . Let X_i^{j+1} be a stratum whose closure intersects X_{j+1}^{j+1} . Then X_i^{j+1} must intersect transversally $W_\epsilon^u(\Lambda_{j+1})$. By the strong transversality condition, X_i^{j+1} must intersect the local unstable manifold of the point into an open subset of Λ_{j+1} . By transitivity of Λ_{j+1} , the closure of X_i^{j+1} contains X_{j+1}^{j+1} . Thus Σ_{j+1} is a stratification.

By local invariance of $(\hat{L}, \hat{\mathcal{L}})$, we endow $L_{X_{j+1}^{j+1}} := \cup_{n \geq 0} f_{|A_{j+1}}^{-n}(D_\epsilon) \cup X_{j+1}^{j+1}$ with the lamination structure given by pulling back $(\hat{L}, \hat{\mathcal{L}})$ and adding the leaves of X_{j+1}^{j+1} . We notice that the transversality of the plaques of \hat{L} is crucial for having still a laminar structure after the last addition of leaves. This is a consequence of the Λ -lemma.

We note that the restriction of $\mathcal{L}_{X_{j+1}^{j+1}}$ to its support minus the one dimensional strata of Σ_{j+1} is a C^1 -foliation on an open subset of M .

For $i \leq j$, let $L_{X_i^{j+1}} := \cup_{n \geq 0} f_{|A_j}^{-n}(L_{X_i^j})$ that we laminate by pulling back $\mathcal{L}_{X_i^j}$.

2. PERSISTENCE OF TRELLIS STRUCTURES

In this section we recall the main theorem of [Ber07] on persistence of stratifications of laminations. The reader should look at this paper for more details and examples.

Let us first recall a few terminologies.

2.1. Plaque-expansiveness. Let (L, \mathcal{L}) be a compact lamination embedded into a manifold M . Let \tilde{f} be a diffeomorphism of M which preserves (L, \mathcal{L}) . For a positive real number ϵ , an ϵ -pseudo-orbit $(x_n)_{n \in \mathbb{Z}}$ which respects \mathcal{L} is a sequence of L such that for all $n \in \mathbb{Z}$, $\tilde{f}(x_n)$ and x_{n+1} belong to a same plaque of \mathcal{L} of diameter less than ϵ . A diffeomorphism \tilde{f} is *plaque-expansive* at (L, \mathcal{L}) if for every small ϵ , for all ϵ -pseudo orbits $(x_n)_n$ and $(y_n)_n$ respecting \mathcal{L} such that x_n and y_n are ϵ -close for every $n \in \mathbb{Z}$, then x_0 and y_0 belong to a same small plaque. This is the definition used in the HPS' Theorem 0.5.

The persistence theorem on trellis structure (see below) works also for endomorphisms of manifold (C^1 -maps that are possibly non invertible). Thus instead

of considering pseudo-orbits, we shall regard *forward pseudo-orbits* which are pseudo-orbits implemented by \mathbb{N} . Moreover the laminations are not all invariant and not all compact. This leads us to generalize the above concepts in the following form.

Let (L, \mathcal{L}) be a lamination. Let f be a continuous map from an open subset V of L into L . Let ϵ be a continuous, positive function on L . An ϵ -*forward-pseudo-orbit* $(x_n)_{n \geq 0} \in V^{\mathbb{N}}$ which respects \mathcal{L} is a sequence such that for all $n \geq 0$, $f(x_n)$ and x_{n+1} belong to a same plaque of \mathcal{L} of diameter less than $\epsilon(x_n)$.

Let $X \subset V$ be a \mathcal{L} -saturated set. We say that f is ϵ -*forward-plaque-expansive at X* if for every pair of ϵ -pseudo-orbits $(x_n)_{n \geq 0} \in V^{\mathbb{N}}$ and $(y_n)_{n \geq 0} \in V^{\mathbb{N}}$ such that $d(x_n, y_n) \leq \epsilon(x_n)$, then the points x_0 and y_0 belong to a same small plaque, which is moreover included in X . We say that f is *forward-plaque-expansive at X* if f is ϵ -forward-plaque-expansive at X for every function ϵ smaller than a given function.

Example 2.1. Let Σ be the stratification constructed in Proposition 1.6 for an AS compact set K of f . Let X be a stratum of Σ and (L_X, \mathcal{L}_X) its tubular neighborhood. By contraction of the plaques of \mathcal{L}_X we can shadow the pseudo-orbits which respect the plaques by an orbit in the same plaques. By normal expansion, there exists a neighborhood V of X such that $f|_V$ is forward plaque-expansive at X .

2.2. Statement of the persistence theorem of stratifications of laminations. Let (A, Σ) be a stratified space endowed with a structure of trellis \mathcal{T} . An *embedding* of (A, Σ, \mathcal{T}) into a manifold M is a homeomorphism from A onto its image in M whose restriction to each tubular neighborhood (L_X, \mathcal{L}_X) is an immersion of laminations. We recall that the set of embeddings of laminations is endowed with the C^1 -compact-open topology (see the introduction). We endow the space of immersions i of (A, Σ, \mathcal{T}) into a manifold M , with the initial topology of the following inclusion into the product of the spaces of embeddings from (L_X, \mathcal{L}_X) into M for every $X \in \Sigma$:

$$i \mapsto (i|_{L_X})_{X \in \Sigma}$$

The initial topology is the coarsest one such that the above map is continuous. Given an open subset $A' \subset A$, we denote by $\Sigma|_{A'}$ the stratification of laminations on A' whose strata are the restrictions $X|_{X \cap A'}$ of the strata $X \in \Sigma$ to $X \cap A'$. Similarly the trellis structure $\mathcal{T}|_{A'}$ is made by restricting each tubular neighborhood.

Theorem 2.2 (Thm 3.1, Rmk 3 [Ber08b]). *Let (A, Σ) be a stratified space supporting a trellis structure \mathcal{T} . Let i be an embedding of (A, Σ, \mathcal{T}) into a manifold M . Let f be a C^1 map from M into itself, which sends the closure of an open subset A' precompact in A into A' .*

We suppose that for every $X \in \Sigma$ there exists an open neighborhood V_X of X in L_X such that:

- i. f preserves and normally expands each stratum X of Σ ,
- ii. f sends each plaque of \mathcal{L}_X included in V_X into a plaque of \mathcal{L}_X ,
- iii. $f|_{V_X}$ is forward plaque-expansive at X , for every $X \in \Sigma$.

Then for f' C^1 -close to f , there exists an embedding i' of $(A', \Sigma|_{A'}, \mathcal{T}|_{A'})$ into M , close to i and a continuous map f'^* C^0 -close to $f^* := f|_{A'}$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & f' & & \\
 & M & \rightarrow & M' & \\
 i' \uparrow & & & & \uparrow i' \\
 & A' & \rightarrow & A' & \\
 & & f'^* & &
 \end{array}$$

Moreover, there exists a family of neighborhoods $(V'_X)_{X \in \Sigma}$ in A' of $(A' \cap X)_{X \in \Sigma}$ respectively such that for every f' C^1 -close to f , the map f'^* sends each plaque of \mathcal{L}_X included in V'_X into the same leaf of \mathcal{L}_X as f^* .

Let us show how to apply this theorem to prove the main theorem. Let (A, Σ) and \mathcal{T} be the stratification of laminations and its trellis structure given by Proposition 1.6 for the AS compact set K . By using Example 2.1, we notice that (A, Σ, \mathcal{T}) satisfies the hypotheses of the above theorem with $A' := f(A)$.

Let $\hat{\Sigma}$ be the stratification of laminations on $\hat{A} := \pi^{-1}(A)$ whose strata are pull backs by π of those of Σ . The leaves of each stratum of $\hat{\Sigma}$ are the connected components of the preimages by π of the leaves of the corresponding stratum of Σ . Let $\hat{\mathcal{T}}$ be the trellis structure on $(\hat{A}, \hat{\Sigma})$ whose tubular neighborhoods are the pull back by π of those of \mathcal{T} . The preservation of $\hat{\Sigma}$ and Property *ii* are clear. The normal expansion of $\hat{X}_i := \pi^{-1}(X_i)$ follows from the normal hyperbolicity of $\hat{\Lambda}_i := \pi^{-1}(\Lambda_i)$ and from the belonging of the leaves of X_i to stable manifolds of Λ_i .

Let us show Property *iii* for some stratum \hat{X}_i . Let V_{X_i} be the neighborhood of X_i in L_{X_i} which satisfies Property *iii* for f . Let $V_{\hat{X}_i}$ be the preimage by π of V_{X_i} .

Let $(\hat{x}_n)_n$ and $(\hat{y}_n)_n$ be two close η -pseudo-orbits of \hat{V}_i which respect the plaques of $\mathcal{L}_{\hat{X}_i}$. Let $(x_n)_n$ and $(y_n)_n$ be the images by π of these pseudo-orbits. For η small enough, x_0 and y_0 belong to a same small plaque of X_i . We can suppose that η is small enough in order that this plaque may be included in a trivialization of $\pi: \hat{M} \rightarrow M$. As \hat{x}_0 and \hat{y}_0 are η -close, included in this trivialization and in the pull back of this plaques, we get that x_0 and y_0 belong to a same small plaque of $\mathcal{L}_{\hat{X}_i}$.

So we can apply Theorem 2.2 with $\hat{A}' := \hat{f}(\hat{A})$. For \hat{f}' close to \hat{f} , we recall that i' denotes the embedding of $(\hat{A}', \hat{\Sigma}|_{\hat{A}'})$ and $(V'_{\hat{X}_i})$ the family of the neighborhoods given by the above theorem.

3. PROOF OF THE MAIN THEOREM

3.1. The settings. We recall the existence of a filtration $(K_j)_j$ of K adapted to $(\Lambda_j)_j$. For every j , let $\hat{K}_j := \pi^{-1}(K_j)$. We construct by decreasing induction, increasing sequences of open subsets $(\hat{O}_j)_{j=1}^N$ of A' and $(O_j)_{j=1}^N$ of \hat{A}' , such that for every j :

- $\hat{O}_j = \pi^{-1}(O_j)$,
- \hat{O}_j has its closure sent into \hat{O}_j by \hat{f}^{-1} ,
- $C_j := cl(\hat{O}_j \setminus \cup_{k>j} \hat{O}_k)$ is included in $V'_{\hat{X}_j}$.

In order to do so we take n_N large enough, then $n_{N-1} \geq n_N$ large enough, \dots , and eventually $n_1 \geq n_2$ large enough such that $(O_j := f^{-n_j}(K \setminus K_{j-1}))_j$ is convenient with $(\hat{O}_j := \pi^{-1}(O_j))_j$.

We put $V_j := \cup_{n \geq 0} \hat{f}^{-n}(C_j)$ which is included in $\hat{f}^{-1}(L_{\hat{X}_j})$ by f^{-1} -stability of L_{X_j} . And so V_j is included in $\hat{f}^{-1}(L_{\hat{X}_j}) \cap \cup_{k \geq j} C_k$. Thus, for every $x \in V_j$, there exists $k \geq j$ s.t. x belongs to $C_k \cap \hat{f}^{-1}(L_{\hat{X}_j})$. Consequently, \hat{f}' sends $i'(x)$ into the image of a $\mathcal{L}_{\hat{X}_k}$ -plaques of $\hat{f}(x) \in L_{\hat{X}_j} \cap L_{\hat{X}_k}$. By compatibility of the tubular neighborhoods, the $\mathcal{L}_{\hat{X}_k}$ -plaque of $\hat{f}(x)$ is included in $\mathcal{L}_{\hat{X}_j}$, and so \hat{f}' sends $i'(x)$ into the image by i of a $\mathcal{L}_{\hat{X}_k}$ -plaque of $\hat{f}(x)$. Therefore, \hat{f}' sends the image by i' of every $\mathcal{L}_{\hat{X}_j}$ -plaques included in V_j into the image by i' of a plaque of $\mathcal{L}_{\hat{X}_j}$.

We recall that \mathcal{L} is the lamination on $\pi^{-1}(K)$ whose leaves are the connected components of the fibers of $\pi|_L$. For $x \in M$, let \mathcal{L}_x be the fiber of $\pi: \hat{M} \rightarrow M$.

For $\delta > 0$, we denote by \mathcal{L}_{jx}^δ the closure of the preimage by π of the $\mathcal{L}_{\hat{X}_j}$ -plaques containing $x \in L_{X_j}$ and included in $\tilde{W}_\delta^s(x)$. Let $\mathcal{L}_{jx}'^\delta$ be the image by i' of \mathcal{L}_{jx}^δ .

A consequence of the λ -contraction of f for the \tilde{d} metric along the stable manifolds, is:

Claim 3.1. For every $\delta > 0$ small enough, $j \in \{1, \dots, N\}$, \hat{f}' sufficiently close to \hat{f} sends $\mathcal{L}_{jx}'^\delta$ into $\mathcal{L}_{j\hat{f}(x)}^\delta$, for every $x \in \pi(C_j)$.

The following lemma will be helpful for the proof:

Lemma 3.2. Let $(x_n)_n \in A^\mathbb{N}$ be a sequence which converges to $x \in K$. Then any sequences $(x'_n)_n \in M^\mathbb{N}$, with $x'_n \in \tilde{W}_\delta^s(x_n)$ has all its limit points in $cl(\tilde{W}_\delta^s(x))$.

Proof. As $\tilde{W}_\delta^s(x)$ is equal to the intersection of $f^{-1}(\tilde{W}_{\lambda\delta}^s(f(x)))$ with the δ -ball centered at x , it is sufficient to prove the lemma when x belongs to $W_\epsilon^s(\Lambda_i)$, for some $i \in \{1, \dots, N\}$. As on $W_\epsilon^s(\Lambda_i)$, the contraction along the stable manifolds is stronger than λ , we get that $\tilde{W}_\delta^s(x)$ is equal to the δ -local stable manifold of x , for any δ sufficiently small. Let us suppose that $(x'_n)_n$ converges to a point x' . We note that x' belongs to the closed δ -ball centered at x . Also x' must have all its orbit δ -close to the one of x . In other words, x' belongs to the closure of the δ -stable manifold of x , which is equal here to $cl(\tilde{W}_\delta^s(x))$. \square

Let N be a smooth section of the Grassmannian of TM C^0 -close to the orthogonal of the tangent space to the fibers of π . Hence we can suppose that N is in direct sum with the tangent space to the fibers of π .

We endow $F := \cup_{y \in \hat{M}} N(y)$ with the canonical vector subbundle structure of $T\hat{M}$. Let α be a positive continuous function small enough such that for every $x \in M$,

$$\{(y, u) \in F : \|u\| < \alpha(x), y \in \mathcal{L}_x\}$$

is embedded by the exponential map \exp associated to the Riemannian metric of \hat{M} .

We define the submersion

$$\begin{aligned} \text{Exp}: F &\rightarrow \hat{M} \\ (y, u) &\mapsto \exp_y \left(\frac{\alpha \circ \pi(y) \cdot u}{\sqrt{1 + \|u\|^2}} \right) \end{aligned}$$

whose restriction to $F_y := N(y)$ is a diffeomorphism onto its open image \mathcal{F}_y , for every $y \in \hat{M}$.

We notice that the restriction of Exp to the zero section of $F|_L$ is equal to the canonical inclusion of \mathcal{L} into \hat{M} .

Let G be the set formed by the submanifolds of \hat{M} C^1 -diffeomorphic to a fiber of π . We endow G with the C^1 -uniform topology. The following claim is an easy consequence of the implicit function theorem.

Claim 3.3. There exists an open neighborhood V_G of $\{(x, \mathcal{L}_x); x \in M\}$ in $M \times G$, such that for every $(x, N) \in V_G$, every $y \in \mathcal{L}_x$, the submanifolds \mathcal{F}_y and N have a transverse intersection which consists of a unique point $I(y, N)$.

Moreover the map I is continuous and its differential with respect to the first variable exists, is injective and depends continuously on $(y, G) \in V_G$.

3.2. Proof by induction. We are going to construct by decreasing induction on $0 \leq i \leq N$, for every $\delta > 0$ sufficiently small and then for every \hat{f}' C^1 -sufficiently close to \hat{f} , a continuous map h_i from \hat{O}_i into \hat{M} such that:

- (a) h_i is an embedding from the lamination $\mathcal{L}|_{\hat{O}_i}$ into M , which is C^1 -close to the canonical inclusion,
- (b) For every $x \in f^{-1}(O_i)$, \hat{f}' sends $h_i(\mathcal{L}_x)$ onto $h_i(\mathcal{L}_{f(x)})$:

$$h_i(\mathcal{L}_{f(x)}) = \hat{f}' \circ h_i(\mathcal{L}_x).$$

- (c) For every $j \geq i$, every $x \in V_j \cap O_i$, the manifold $h_i(\mathcal{L}_x)$ is included in $\mathcal{L}'_{jx}{}^\delta$.
- (d) For every $y \in \hat{O}_i$, $h_i(y)$ belongs to \mathcal{F}_y .

We notice that the main theorem is proved as soon as this induction is accomplished.

Remark 3.4. By (a), (b) and (d), for every $y \in \hat{O}_i$, the point $h_i(y)$ is equal to $I(y, \hat{f}'(h_i(\mathcal{L}_{f^{-1}(x)})))$ with $x = \pi(y)$.

Let us now proceed to the proof of the induction.

Along the induction, δ is going to be supposed smaller and then \hat{f}' is going to be supposed closer to \hat{f} .

3.2.1. *Step $i = N$.* For $\beta > 0$ small enough, the β -local unstable manifolds $(W_\beta^u(x))_{x \in \Lambda_N}$ form plaques of a lamination on $W_\beta^u(\Lambda_N)$. For some $k \geq 0$, $f^k(W_\beta^u(\Lambda_N))$ covers $cl(O_N)$. Also the local unstable manifolds in $f^k(W_\beta^u(\Lambda_N))$ are plaques of a lamination normally expanded by f^{-1} . By normal hyperbolicity of $\hat{\Lambda}_N$, the backward image $\hat{\mathcal{L}}^u$ of this lamination by π is normally expanded by \hat{f}^{-1} and plaque-expansive. Thus $\hat{\mathcal{L}}^u$ is persistent (by using Theorem 2.2 with a stratification consisting of a single stratum or see [Ber08a] to deal only with laminations). In other words there exist $\gamma < \beta$ and an immersion i^u of $\hat{\mathcal{L}}^u$ into \hat{M} close to the canonical inclusion such that we have for \hat{f}' C^1 -close to \hat{f} :

$$\hat{f}'(i^u(\hat{\mathcal{L}}_x^{u\gamma})) \subset i^u(\hat{\mathcal{L}}_{f(x)}^{u\beta}), \quad \forall x \in f^{-1}(O_N),$$

with $\hat{\mathcal{L}}_x^{u\beta}$ and $\hat{\mathcal{L}}_x^{u\gamma}$ are the β and γ -neighborhoods of \mathcal{L}_x in its leaf of $\hat{\mathcal{L}}^u$.

For γ , β , and δ sufficiently small, and then \hat{f}' sufficiently close to \hat{f} , for every $x \in O_N$, the submanifolds $i^u(\hat{\mathcal{L}}_x^{u\gamma})$ and $i^u(\hat{\mathcal{L}}_x^{u\beta})$ intersect \mathcal{L}'_{Nx} transversally at a unique submanifold \mathcal{L}'_x C^1 -close to \mathcal{L}_x . For $y \in \mathcal{L}_x$, let $h_N(y)$ be the intersection point of the transverse intersection of \mathcal{F}_y with \mathcal{L}'_x .

We notice that the map h_N is an immersion of the lamination $\mathcal{L}|_{\hat{O}_N}$ into \hat{M} close to the canonical inclusion, for \hat{f}' close to \hat{f} . As i' and i^u are embeddings of laminations, h_N is also an embedding.

We recall that for every $x \in f^{-1}(O_N)$, the submanifold \mathcal{L}'_{Nx} is sent by \hat{f}' into $\mathcal{L}'_{Nf(x)}$ and the submanifold $i^u(\mathcal{L}_x^{u\gamma})$ is sent by \hat{f}' into $i^u(\mathcal{L}_{f(x)}^{u\beta})$. Consequently \hat{f}' sends $\mathcal{L}'_x = \mathcal{L}'_{Nx} \cap i^u(\mathcal{L}_x^{u\gamma})$ into $\mathcal{L}'_{f(x)} := \mathcal{L}'_{Nf(x)} \cap i^u(\mathcal{L}_{f(x)}^{u\beta})$. Moreover as \hat{f}' is a diffeomorphism and the submanifold \mathcal{L}'_x is compact, by connectedness, $\hat{f}'(\mathcal{L}'_x)$ is equal to $\mathcal{L}'_{f(x)}$.

3.2.2. *Step $i + 1 \rightarrow i$.* Let us assume the induction hypothesis for $i + 1 \leq N$.

By normal hyperbolicity of $\hat{\Lambda}_i$ and local invariance of \mathcal{L}'_i , for every $\sigma > 0$ small enough, there exists a \mathcal{L} -saturated neighborhood \hat{W}_i of $\hat{\Lambda}_i$ in V_i such that for \hat{f}' close enough to \hat{f} , the cone field C''^σ whose cone at $y \in i'(\hat{W}_i)$ is $C''^\sigma(y) := \{v \in T\mathcal{L}'_i : \angle(v, T_{\pi(y)}\mathcal{L}) < \sigma\}$ satisfies the following properties:

(1) for every $y \in i'(\hat{W}_i) \cap \hat{f}'^{-1}(i'(\hat{W}_i))$:

$$cl\left(T\hat{f}'(C''^\sigma(y))\right) \subset C''^\sigma(\hat{f}'(y)) \cup \{0\}$$

(2) the intersection between $C''^\sigma(y)$ and F_y is empty.

We fix for this step δ , σ and then \hat{W}_i small enough in order that the following claim is satisfied.

Claim 3.5. For every \hat{f}' C^1 -close to \hat{f} , we have:

- For every $x \in \pi(\hat{W}_i)$, \hat{f}' sends $\mathcal{L}_{ix}'^\delta$ into $\mathcal{L}_{if(x)}'^\delta$.
- For $x \in \pi(\hat{W}_i)$ and $h \in C^1(\mathcal{L}_x, \mathcal{L}_{ix}'^\delta)$, if $Th(T\mathcal{L}_x)$ is included in C'^σ , then $(x, h(\mathcal{L}_x))$ belongs to V_G .
- There exists a neighborhood $\hat{W}'_i \subset \hat{W}_i$ of $\hat{\Lambda}_i$ such that for every $y \in \hat{W}'_i$, $x := \pi(y)$, the tangent space of $\mathcal{F}_y \cap \mathcal{L}_{ix}'^\delta$ is included in the closure of the complement of C'^σ , although $\mathcal{L}_i'^\delta$ is included in W_i .

We remind that the filtration $(K_j)_j$ satisfies $\Lambda_i := \bigcap_{n \in \mathbb{Z}} f^n(K_i \setminus K_{i-1}) = \bigcap_{n \in \mathbb{Z}} f^n(O_i \setminus O_{i+1})$. As each O_j is sent into itself by the inverse of the dynamics, we fix a large $M \geq 0$ such that $\pi^{-1}(f^{-M}(O_i) \setminus f^M(O_{i+1}))$ has a closure included in \hat{W}_i .

Let \hat{f}' be close enough to \hat{f} , such that the following map is well defined:

$$\tilde{h}_{i+1} : y \in \tilde{O}_{i+1} \mapsto I\left(y, \hat{f}'^{m+1} \circ h_{i+1}(\mathcal{L}_{f^{-n-1} \circ \pi(y)})\right) \\ \text{with } \tilde{O}_{i+1} := \hat{f}^{M+1}(\hat{O}_{i+1}) \cap \hat{O}_i.$$

By induction hypothesis (b), for $x \in O_{i+1}$, the submanifolds $h_{i+1}(\mathcal{L}_x)$ and $\hat{f}'^{M+1} \circ h_{i+1}(\mathcal{L}_{f^{-M-1}(x)})$ are equal. By induction hypothesis (d), the map \tilde{h}_{i+1} is equal to h_{i+1} on \hat{O}_{i+1} . Moreover, one easily check that:

$$(1) \quad \tilde{h}_{i+1}(y) = I\left(y, \hat{f}' \circ \tilde{h}_{i+1}(\mathcal{L}_{f^{-1}(x)})\right), \quad \text{for } y \in \tilde{O}_{i+1}, x := \pi(y).$$

Let us show that for \hat{f}' close enough to \hat{f} , the point $\tilde{h}_{i+1}(y)$ belongs to $\mathcal{L}_{j\pi(y)}'^\delta$ for every $j < i$ and $y \in V_j \cap \tilde{O}_{i+1}$. By \hat{f}^{-1} -stability of $V_j \cap \tilde{O}_{i+1}$, the $n+1$ first preimages of y by \hat{f} belong also to V_j .

For \hat{f}' close enough to \hat{f} , the submanifold $h_{i+1}(\mathcal{L}_{f^{-n-1}(x)})$ is close to $i'(\mathcal{L}_{f^{-n-1}(x)})$ and so $\hat{f}'^{m+1} \circ h_{i+1}(\mathcal{L}_{f^{-n-1}(x)})$ is included in $\mathcal{L}_{jx}'^\delta$, for every $y \in V_i \cap \tilde{O}_{i+1}$ and $x := \pi(y)$. Thus $\tilde{h}_{i+1}(y)$ belongs to $\mathcal{L}_{jx}'^\delta$, for every $j < i$ such that $y \in V_j \cap \tilde{O}_{i+1}$.

Consequently, for every $y \in \tilde{O}_{i+1} \cap \hat{W}_i$, $x := \pi(y)$, by compatibility of the tubular neighborhood, the point $\tilde{h}_{i+1}(y)$ belongs to $\mathcal{L}_{ix}'^\delta$. Thus, for \hat{f}' close enough to \hat{f} , by (a), the space $T\tilde{h}_{i+1}(T_y \mathcal{L}_x)$ is included in $cl(C'^\sigma(\tilde{h}_{i+1}(y)))$. As for every $m \geq 0$, the intersection $\hat{f}^{-M}(\hat{O}_i)$ is a subset of $\hat{W}'_i \cup \tilde{O}_{i+1}$, the following sequence is well defined:

$$h^{(0)} := \tilde{h}_{i+1}$$

$$h^{(m+1)} := y \in \hat{f}^{m+1}(\tilde{O}_{i+1}) \cap \hat{f}^{-M}(\hat{O}_i) \mapsto I\left(y, \hat{f}' \circ h^{(m)}(\mathcal{L}_{f^{-1}(x)})\right), \quad \text{with } x = \pi(y).$$

We note that for $m \geq 0$, the map $h^{(m)}$ is an immersion of $\mathcal{L}_{|\hat{f}^m(\tilde{O}_{i+1}) \cap \hat{f}^{-M}(\hat{O}_i)}$ into \hat{M} , which satisfies:

$$h^{(m)}(y) = I\left(y, \hat{f}' \circ h^{(m)}(\mathcal{L}_{f^{-1}(x)})\right), \quad \forall y \in \hat{f}^m(\tilde{O}_{i+1}) \cap \hat{f}^{-M}(\hat{O}_i), \text{ with } x := \pi(y).$$

Moreover for all $m' \geq m \geq 0$ and $y \in \hat{f}^m(\tilde{O}_{i+1}) \cap \hat{f}^{-M}(\hat{O}_i)$, the points $h^{(m')}(y)$ and $h^{(m)}(y)$ are equal.

Let $W^u(\hat{\Lambda}_i) := \pi^{-1}(W_K^u(\Lambda_i))$. We define h_i on $\hat{f}^{-M}(\hat{O}_i) \setminus W^u(\hat{\Lambda}_i)$ by:

$$h_i(y) = h^{(m)}(y), \quad \text{if } y \in \hat{f}^m(\tilde{O}_{i+1}) \cap \hat{f}^{-M}(\hat{O}_i)$$

We remark that h_i is an immersion of $\mathcal{L}_{|\hat{f}^{-M}(\hat{O}_i) \setminus W^u(\hat{\Lambda}_i)}$ into M . Moreover h_i satisfies:

$$(2) \quad \begin{cases} h_i(y) = I(y, \hat{f}' \circ h_i(\mathcal{L}_{f^{-1}(x)})) & \text{for } y \in \hat{f}^{-M}(\hat{O}_i) \setminus W^u(\hat{\Lambda}_i), \ x := \pi(y), \\ h_i(y) \in \mathcal{L}_{ix}^{\delta} & \text{for } y \in \hat{f}^{-M}(\hat{O}_i) \cap V_i \setminus W^u(\hat{\Lambda}_i). \end{cases}$$

For \hat{f}' sufficiently close to \hat{f} , we can extend h_i on $\hat{O}_i \setminus W^u(\hat{\Lambda}_i)$ via the expression:

$$h_i(y) = I(y, (\hat{f}'^M \circ h_i)(\mathcal{L}_{f^{-M}(x)})), \quad \text{with } x = \pi(y).$$

To define h_i on $\hat{O}_i \cap W^u(\hat{\Lambda}_i)$, we proceed similarly as in $i = N$: the image by h_i of \mathcal{L}_x is the intersection of $\mathcal{L}_{ix}^{\delta}$ with a the persistent preimage by π of a local unstable manifold of x .

Proof that h_i is an immersion. It remains only to prove the continuity of h_i and the continuity of its differential with respect to $T\mathcal{L}$. Moreover, we only need to show this on $W^u(\hat{\Lambda}_i) \cap \hat{O}_i$. By Remark 3.4, this is equivalent to show this at $W^u(\hat{\Lambda}_i) \cap \hat{W}'_i$.

Let us begin with the proof of the continuity of h_i . We suppose for the sake of contradiction the existence of a sequence $(y_n)_n \in \hat{W}_i^N$ which converges to $y \in W^u(\hat{\Lambda}_i) \cap \hat{W}'_i$ but such that $(h_i(y_n))_n$ does not converge to $h_i(y)$.

By local compactness of M , we may suppose that $(h_i(y_n))_n$ converges to z different to $h_i(y)$. Put $x_n := y_n$ and $x := \pi(y)$. By induction hypothesis (c), the point $h_i(y)$ belongs to $\mathcal{L}_{ix}^{\delta}$ and the point $h_i(y_n)$ belongs to $\mathcal{L}_{ix_n}^{\delta}$. By Lemma 3.2, the point z belongs to $\mathcal{L}_{ix}^{\delta}$. Also since

$$f^{-1}(\pi(V_i) \cap W_K^u(\Lambda_i)) \subset \pi(V_i) \cap W_K^u(\Lambda_i),$$

we have for every $k \geq 0$ and then n sufficiently large the points $f^{-k}(x_n), f^{-k}(x)$ that belong to $\pi(V_i)$ and so by Lemma 3.2 the points $\hat{f}'^{-k} \circ h_i(y)$ and $\hat{f}'^{-k}(z) = h_i \circ \hat{f}^{-k}(y)$ belong to $\mathcal{L}_{if^{-k}(x)}^{\delta}$.

On the other hand, $\hat{f}'^{-k} \circ h_i(y)$ and $\hat{f}'^{-k}(z)$ belong to $\mathcal{F}_{\hat{f}'^{-k}(y)}$. Also these points belong to \hat{W}'_i for k sufficiently large. Thus we can link $\hat{f}'^{-k} \circ h_i(y)$ and $\hat{f}'^{-k}(z)$ by a C^1 -path included in $\mathcal{F}_y \cap \mathcal{L}_{if^{-k}(x)}^{\delta}$.

By the last proposition of the claim, the regarded path has its tangent space included in the closure of the complement of C''^{σ} . This implies that its preimages by \hat{f}' have also their tangent space included in the closure of the complement of C''^{σ} and that their length increase exponentially fast. This is a contradiction

with the fact that $\hat{f}'^{-k}(z)$ and $\hat{f}'^{-k} \circ h_i(y)$ belong to $\mathcal{L}'_{if^{-k}(x)}^\delta$, for every k large. Thus h_i is continuous on \hat{O}_i .

Let us show by the sake of contradiction the continuity of the derivative of h_i with respect to $T\mathcal{L}$ at $W^u(\hat{\Lambda}_i) \cap \hat{W}'_i$. Let $(y_n)_n \in \hat{W}'N_i$ which converges to $y \in W^u(\hat{\Lambda}_i) \cap \hat{W}'_i$, such that $(Th_i(y_n))_n$ does not converge to $Th_i(y)$. As $Th_i(T_{y_n}\mathcal{L})$ is included in C'^σ for every $n \geq 0$, we can suppose that $(Th_i(T_{y_n}\mathcal{L}))_n$ converges to a d -plane $P' \subset cl(C'^\sigma)$ different of $P := Th_i(T_y\mathcal{L}) \subset cl(C'^\sigma)$. For the same reasons as before, $T\hat{f}'^{-k}(P)$ and $T\hat{f}'^{-k}(P')$ are included in $C'^\sigma(\hat{f}'^{-k} \circ h_i(y))$ for every k large. By the projective hyperbolicity of the cone field C'^σ , we get a contradiction.

h_i is an embedding. By induction hypotheses (b) and (d) it is sufficient to show that $h_i|_{\hat{f}^{-1}(\hat{O}_i)}$ is an embedding. For this end, it is sufficient to show that h_i is injective since $\hat{f}^{-1}(\hat{O}_i)$ is precompact in \hat{O}_i .

Also by induction hypothesis (d), it is sufficient to prove that two different leaves of $\mathcal{L}|_{\hat{O}_i}$ have disjoint images by h_i .

We remind that:

$$(3) \quad \forall x \in O_i, \quad \hat{f}'^{-n} \circ h_i(\mathcal{L}_x) = h_i(\mathcal{L}_{f^{-n}(x)})$$

Let $x, x' \in O_i$ be such that $h_i(\mathcal{L}_x)$ intersects $h_i(\mathcal{L}_{x'})$.

If neither x nor x' belongs to $W_K^u(\Lambda_i)$. Then there exists $n \geq 0$ such that $f^{-n}(x)$ and $f^{-n}(x')$ belong to O_{i+1} . By (3), the image by h_i of $\mathcal{L}_{f^{-n}(x)}$ intersects the one of $\mathcal{L}_{f^{-n}(x')}$. Thus by induction, $f^{-n}(x)$ and $f^{-n}(x')$ are equal and so x equals x' .

If x belongs to $W_K^u(\Lambda_i)$ then the preorbit of \mathcal{L}_x lands to $\hat{\Lambda}_i$. If x' does not belong to $W_K^u(\Lambda_i)$, then for n large enough, $\hat{f}'^{-n}(\mathcal{L}_x)$ is far from $\hat{\Lambda}_i$. As h_i is C^0 -close to the canonical inclusion inclusion, $h_i \circ \hat{f}'^{-n}(\mathcal{L}_x)$ and $h_i \circ \hat{f}'^{-n}(\mathcal{L}_{x'})$ are disjoint. By (3), this is a contradiction.

It only remains to proof the injectivity of h_i on $W^u(\hat{\Lambda}_i) \cap \hat{O}_i$. In order to do so we proceed as for $i = N$.

REFERENCES

- [Ber07] P. Berger. Persistence of stratification of normally expanded laminations. *arXiv:math.DS*, 2007.
- [Ber08a] P. Berger. Persistence of laminations. *arXiv:math.DS*, 2008.
- [Ber08b] Pierre Berger. Persistence of stratifications of normally expanded laminations. *C. R. Math. Acad. Sci. Paris*, 346(13-14):767–772, 2008.
- [dM73] W. de Melo. Structural stability of diffeomorphisms on two-manifolds. *Invent. Math.*, 21:233–246, 1973.
- [Ghy99] É. Ghys. Laminations par surfaces de Riemann. In *Dynamique et géométrie complexes (Lyon, 1997)*, volume 8 of *Panor. Synthèses*, pages ix, xi, 49–95. Soc. Math. France, Paris, 1999.
- [HPPS70] M. Hirsch, J. Palis, C. Pugh, and M. Shub. Neighborhoods of hyperbolic sets. *Invent. Math.*, 9:121–134, 1969/1970.

- [HPS77] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 583.
- [Mañ88] R. Mañé. A proof of the C^1 stability conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (66):161–210, 1988.
- [Mat73] J. N. Mather. Stratifications and mappings. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 195–232. Academic Press, New York, 1973.
- [PS70] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 223–231. Amer. Math. Soc., Providence, R.I., 1970.
- [Rob71] J. W. Robbin. A structural stability theorem. *Ann. of Math. (2)*, 94:447–493, 1971.
- [Rob76] C. Robinson. Structural stability of C^1 diffeomorphisms. *J. Differential Equations*, 22(1):28–73, 1976.
- [Shu78] M. Shub. *Stabilité globale des systèmes dynamiques*, volume 56 of *Astérisque*. Société Mathématique de France, Paris, 1978. With an English preface and summary.
- [Sma67] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

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